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EXTENSIONS OF LIE-RINEHART ALGEBRAS AND THE CHERN-WEIL CONSTRUCTION

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ABSTRACT. A Chern-Weil construction for extensions of Lie-Rinehart algebras is introduced. This generalizes the classical Chern-Weil construction in differential geometry and yields characteristic classes for arbitrary extensions of Lie-Rinehart algebras. Some examples arising from spaces with singularities and from foliations are given that cannot be treated by means of the classical Chern-Weil construction.

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Introduction

Let R be a commutative ring, and let A be a commutative R -algebra. We introduce a general Chern-Weil construction which yields characteristic classes for extensions of (R, A) -Lie algebras. These (R, A) -Lie algebras have been introduced by HERZ [10] under the name “pseudo-algèbre de Lie” and were examined by PALAIS [28] under the name “ d -Lie ring” and thereafter by RINEHART [33], who introduced the terminology “ (R, A) -Lie algebra”. An (R, A) -Lie algebra is a Lie algebra L over the ground ring R , together with an action of L on A and an A -module structure on L , and the two structures satisfy suitable compatibility conditions which generalize the usual properties of the Lie algebra of smooth vector fields on a smooth manifold viewed as a module over its ring of smooth functions; a precise definition will be reproduced in Section 1 below. With a suitable notion of morphism, such pairs (A, L) constitute a category, and we shall refer to such a pair as a *Lie-Rinehart algebra*.

A principal bundle gives rise to an extension of Lie-Rinehart algebras which arise as spaces of sections of an extension of vector bundles, introduced by ATIYAH [2] and now usually called the *Atiyah sequence* of the principal bundle; see (2.2) below for details. It is also common to talk about *transitive Lie algebroids*, cf. [1, 21, 29–32]. PRADINES [29, 30] in fact introduced the more general concept of (not necessarily transitive) Lie algebroid, but the general notion does not involve an extension in the sense studied in this paper. ALMEIDA AND MOLINO [1] have shown that Lie’s third theorem does not hold for transitive Lie algebroids: *not every transitive Lie algebroid integrates to a principal bundle*. This provides a negative answer to a question raised by Pradines. MACKENZIE [21] developed obstructions for the integrability of Lie algebroids.

For a principal bundle, our Chern-Weil construction boils down to the usual Chern-Weil construction [3], [7], [17], [27]; details will be given in Section 3 below. However, there are interesting examples, e. g. arising from foliations or from quantization problems, that do not come from a principal bundle; we shall describe some such examples in Section 4 below. A Chern-Weil homomorphism for a transitive Lie algebroid has been set up by TELEMAN [37], and our construction extends that of Teleman as well. The “transverse Chern-Weil map” for equivariant principal bundles constructed in [22] provides a Chern-Weil theory for extensions of principal bundles.

We now give a brief overview of the contents of the paper: In Section 1 we recall some of the basic notions. In Section 2 we generalize the usual concepts of connection and curvature in a principal bundle to arbitrary extensions of Lie-Rinehart algebras. Within the category of finite rank smooth vector bundles, these notions have already been rephrased by MACKENZIE [21] in the language of Atiyah sequences and transitive Lie algebroids. In Section 3 we introduce a Chern-Weil construction for an extension

$$(0.1) \quad 0 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 0$$

of Lie-Rinehart algebras under the assumption that the extension splits in the category of A -modules. This Chern-Weil construction furnishes a morphism

$$(0.2) \quad \mathrm{Hom}_A(\Sigma'_A[s^2L'], A)^L \longrightarrow \mathrm{Alt}_A(L'', A)$$

of differential graded commutative R -algebras whose induced morphism on homology depends only on the congruence class of the extension (0.1); see (3.8.1) and (3.8.2) below for details. Here s^2L' refers to the double suspension of L' , and $\Sigma'_A[s^2L']$ is the *symmetric coalgebra* on s^2L' over A , so that its dual $\text{Hom}_A(\Sigma'_A[s^2L'], A)$ is a graded commutative algebra; furthermore, as usual the notation $-^L$ indicates the invariants with respect to the induced L -action, and the source $\text{Hom}_A(\Sigma'_A[s^2L'], A)^L$ is equipped with the zero differential. The morphism (0.2) involves the notion of curvature for an extension of the kind (0.1), and in order for this curvature to be defined, the underlying extension of A -modules must split. When L' is finitely generated and free as an A -module, the graded algebra $\text{Hom}_A(\Sigma'_A[s^2L'], A)$ is just the polynomial A -algebra on an A -basis of the dual of s^2L' but the above description works *without any finiteness assumption*.

In view of its complete generality, our approach is likely to prove useful for geometrical systems in infinite dimensions and, furthermore, for systems with singularities where e. g. “smooth” functions are to be understood in the sense of WHITNEY [38], [39], see Section 4 for details. While *in the classical finite dimensional case without singularities there was no need to distinguish* between e. g. formal differentials and differential forms and hence not between an exterior algebra $\Lambda_A\mathfrak{g}^*$ over the dual of a Lie algebra \mathfrak{g} over an algebra A and an algebra $\text{Alt}_A(\mathfrak{g}, A)$ of differential forms and likewise, between a symmetric algebra $\Sigma_A\mathfrak{g}^*$ and a corresponding algebra $\text{Hom}_A(\Sigma'_A[\mathfrak{g}], A)$ of forms etc., under more general circumstances when the requisite A -modules are no longer projective or not even reflexive more care is needed and the appropriate objects to work with are those involving A -valued forms etc. This claim is well illustrated by our Chern-Weil map where the *true* algebra of characteristic classes for an extension (0.1) of Lie-Rinehart algebras is an algebra of the kind $\text{Hom}_A(\Sigma'_A[s^2L'], A)^L$ rather than the symmetric algebra on the dual of s^2L' . See in particular what is said at the beginning of Section 3.

I am much indebted to K. Mackenzie for a number of most valuable comments on a draft of the paper. It is a pleasure to dedicate this paper to Jim Stasheff; it has in fact been influenced by his work on characteristic classes [25] and on differential homological algebra, cf. e. g. [15]. Furthermore, about ten years ago, studying his paper [34], I encountered Lie-Rinehart algebras for the first time and thereafter discovered their significance for Poisson structures [11, 12]. Since then Jim encouraged me to study these notions per se, followed my investigations and generously offered support. Lie-Rinehart algebras and strong homotopy generalizations thereof also arose in his work on homological perturbations in the BRST-description of constrained hamiltonian systems and variants thereof; for these matters, see e. g. his papers [34–36]. In this area, there is still a huge unexplored territory.

1. Lie-Rinehart algebras and their modules

We review briefly the concept of Lie-Rinehart algebras. We then give descriptions of appropriate module-, algebra-, coalgebra-, Lie algebra-, etc. structures over Lie-Rinehart algebras and spell out the corresponding differential graded objects.

Let R be a commutative ring, fixed throughout; the unadorned tensor product symbol \otimes will always refer to the tensor product over R . Further, let A be a commutative R -algebra. An (R, A) -Lie algebra [33] is a Lie algebra L over R which acts on (the left of A) by derivations (written $(\alpha \otimes a) \mapsto \alpha a$), and is also an A -module (the structure map being written $(a \otimes \alpha) \mapsto a\alpha$), in such a way that suitable compatibility conditions are satisfied which generalize the usual properties of the Lie algebra of vector fields on a smooth manifold viewed as a module over its ring of functions; these conditions read

$$(1.1.a) \quad (a\alpha)(b) = a(\alpha(b)), \quad \alpha \in L, a, b \in A,$$

$$(1.1.b) \quad [\alpha, a\beta] = a[\alpha, \beta] + \alpha(a)\beta, \quad \alpha, \beta \in L, a \in A.$$

When the emphasis is on the pair (A, L) , with the mutual structure of interaction, we refer to a *Lie-Rinehart algebra*. Given two Lie-Rinehart algebras (A, L) and (A', L') , a *morphism* $(\phi, \psi): (A, L) \longrightarrow (A', L')$ of *Lie-Rinehart algebras* is the obvious thing, that is, ϕ and ψ are morphisms in the appropriate categories that are compatible with the additional structure. With this notion of morphism, Lie-Rinehart algebras constitute a category. Apart from the example of smooth functions and smooth vector fields on a smooth manifold, a related (but more general) example is the pair consisting of a commutative algebra A and the R -module $\text{Der}(A)$ of derivations of A with the obvious A -module structure; here the commutativity of A is crucial.

Let L be an (R, A) -Lie algebra. An R -module M having the structures of a left A -module and that of a left L -module $\omega: L \rightarrow \text{End}(M)$ is said to be an (A, L) -module provided the actions are compatible, that is to say, for $\alpha \in L, a \in A, m \in M$,

$$(1.2.a) \quad (a\alpha)(m) = a(\alpha(m)),$$

$$(1.2.b) \quad \alpha(am) = a\alpha(m) + \alpha(a)m.$$

For example, given a smooth manifold, such a structure with respect to the Lie algebra of smooth vector fields on the module of smooth sections of a smooth vector bundle is just a flat connection.

Let L be an (R, A) -Lie algebra and M an (A, L) -module. The R -multilinear alternating functions from L into M with the CARTAN-CHEVALLEY-EILENBERG [6] differential d given by

$$(1.3) \quad \begin{aligned} (df)(\alpha_1, \dots, \alpha_n) = & \quad (-1)^n \sum_{i=1}^n (-1)^{(i-1)} \alpha_i(f(\alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_n)) \\ & + \quad (-1)^n \sum_{j < k} (-1)^{(j+k)} f([\alpha_j, \alpha_k], \alpha_1, \dots, \widehat{\alpha}_j, \dots, \widehat{\alpha}_k, \dots, \alpha_n) \end{aligned}$$

constitute a chain complex $\text{Alt}_R(L, M)$ where as usual ‘ $\widehat{}$ ’ indicates omission of the corresponding term. The sign $(-1)^n$ in (1.3) has been introduced according to the

usual Eilenberg-Koszul convention in differential homological algebra for consistency with (1.3') below and with what is said in our follow up paper [13]; in the classical approach such a sign does not occur. As observed first by PALAIS [28], the differential d on $\text{Alt}_R(L, M)$ passes to an R -linear differential on the graded A -submodule $\text{Alt}_A(L, M)$ of A -multilinear functions, written d , too; this differential will not be A -linear unless L acts trivially on A , though. Before we proceed further we mention that a *distinction* between *graded* A -objects and *differential graded* R -objects will persist throughout. We shall carry out most constructions, e. g. coalgebras, algebras, etc. over A ; however, in view of the non-triviality of the action of L on A , most resulting differential graded objects will be over the ground ring R only.

Given (A, L) -modules M' and M'' , the usual formula

$$(1.4.1) \quad \alpha(x \otimes_A y) = \alpha(x) \otimes_A y + x \otimes_A \alpha(y), \quad \alpha \in L, x \in M', y \in M'',$$

endows the tensor product $M' \otimes_A M''$ with the structure of an (A, L) -module, referred to as the *tensor product of M' and M'' in the category of (A, L) -modules*; if M is another L -module, a pairing $\mu_A: M' \otimes_A M'' \rightarrow M$ of A -modules which is a morphism of (A, L) -modules (with respect to (1.4.1)) will be said to be a *pairing of (A, L) -modules*. Given such a pairing μ_A of (A, L) -modules, let

$$\mu = \mu_A \pi: M' \otimes_R M'' \rightarrow M' \otimes_A M'' \rightarrow M$$

be the indicated pairing of L -modules, and *define* the shuffle multiplication of R -multilinear, alternating maps by

$$(1.4.2) \quad \begin{aligned} & (\alpha \wedge \beta)(x_1, \dots, x_{p+q}) \\ &= (-1)^{|\alpha||\beta|} \sum_{\sigma} \text{sign}(\sigma) \mu(\alpha(x_{\sigma(1)}, \dots, x_{\sigma(p)}) \otimes \beta(x_{\sigma(p+1)}, \dots, x_{\sigma(p+q)})), \end{aligned}$$

where $\alpha \in \text{Alt}_R^p(L, M')$, $\beta \in \text{Alt}_R^q(L, M'')$, $x_1, \dots, x_{p+q} \in L$. This yields a pairing

$$(1.5) \quad \wedge: \text{Alt}_R(L, M') \otimes \text{Alt}_R(L, M'') \rightarrow \text{Alt}_R(L, M)$$

of chain complexes which is associative in the obvious sense; here σ runs through (p, q) -shuffles and $\text{sign}(\sigma)$ refers to the sign of σ . The sign $(-1)^{|\alpha||\beta|}$ in the formula (1.4.2) does usually *not* occur in the descriptions given in the literature. This sign is dictated by the graded tensor product in differential homological algebra, and since we shall have occasion to extend the pairing (1.5) to the differential graded context we must insist on this sign. See (1.6.5) below for details. The pairing (1.5) induces a pairing

$$(1.5') \quad \wedge: \text{Alt}_A(L, M') \otimes_R \text{Alt}_A(L, M'') \rightarrow \text{Alt}_A(L, M)$$

of chain complexes over R , still denoted by \wedge . In particular, the graded commutative A -algebra $\text{Alt}_A(L, A)$ inherits a structure of a differential graded commutative algebra over the ground ring R but *not* over A unless L acts trivially on A . The pairing \wedge plainly factors through a pairing

$$(1.5'') \quad \text{Alt}_A(L, M') \otimes_A \text{Alt}_A(L, M'') \rightarrow \text{Alt}_A(L, M)$$

of graded A -modules, uniquely determined by the given data, but (1.5'') will *not* be compatible with the differentials unless L acts trivially on A . Another description of the pairing (1.5'') will be given in (1.6.5) below.

A conceptual explanation of these facts in terms of standard homological algebra over a suitable universal algebra is due to RINEHART [33] and has been elaborated upon in our paper [11] to which we refer for details. Here we only mention that any Lie-Rinehart algebra (A, L) determines a universal R -algebra $U(A, L)$; for example, when A is the algebra of smooth functions on a smooth manifold N and L the Lie algebra of smooth vector fields on N , then $U(A, L)$ is the *algebra of* (globally defined) *differential operators on N* . For an (A, L) -module M , the *cohomology of L with coefficients in M* is then defined by

$$H_A^*(L, M) = \text{Ext}_{U(A, L)}^*(A, M),$$

cf. [11, 33]. When L is projective as an A -module, the chain complex $(\text{Alt}_A(L, M), d)$ (reproduced above) computes this cohomology; in the present paper we shall exclusively work with the chain complex $(\text{Alt}_A(L, M), d)$, whether or not L is projective as an A -module.

1.6. MORE STRUCTURE.

Let L be an (R, A) -Lie algebra, and let M be an (A, L) -module, with structure map $\omega: L \rightarrow \text{End}(M)$. At times we shall assume M equipped with additional structure, e. g. that of an algebra, a chain complex, a Lie algebra, etc. Chain complexes with a non-zero differential will not explicitly occur as (A, L) -modules, though; all we shall need are graded (A, L) -modules but we can handle chain complexes at no extra cost and hence we shall do so. The material to be given until the end of this Section is mostly folk-lore; since it is difficult to give precise references, we explain some of the requisite details. We note that A could be just the ground ring R , but in view of later applications it will be convenient to distinguish carefully between A and R .

1.6.1. LIE ALGEBRAS OVER A . Let $M = \mathfrak{g}$ be a Lie algebra over A , with Lie bracket $[\cdot, \cdot]: \mathfrak{g} \otimes_A \mathfrak{g} \rightarrow \mathfrak{g}$. We shall say that \mathfrak{g} is an (A, L) -Lie algebra if the structure map $[\cdot, \cdot]$ is a morphism of (A, L) -modules or, what amounts to the same, if the values of ω lie in $\text{Der}(\mathfrak{g}) \subseteq \text{End}(\mathfrak{g})$; we shall then occasionally write $\omega: L \rightarrow \text{Der}(\mathfrak{g})$. For an (A, L) -Lie algebra \mathfrak{g} , the pairing (1.5') with respect to $\mu_A = [\cdot, \cdot]$ endows the A -multilinear forms $\text{Alt}_A(L, \mathfrak{g})$ with values in \mathfrak{g} with the structure of a differential graded Lie algebra over R . In the special case where $A = R$ with trivial action this is of course well known.

1.6.2. CHAIN COMPLEXES OVER A . Let $M = C$ be a chain complex over A , and write $Z^0(\text{End}(C))$ for the (homogeneous) chain maps from C to itself (of degree zero), i. e. for the cycles in the corresponding Hom-complex. We shall say that C is an (A, L) -chain complex provided the action ω factors through an action $\omega: L \rightarrow Z^0(\text{End}(C))$ (denoted still by ω with a slight abuse of notation). For an (A, L) -chain complex C , the differential (1.3) endows the A -multilinear forms $\text{Alt}_A(L, C)$ with values in C with the structure of a chain complex over R , but some more care is needed to explain what this really means: Write

$$d^0: \text{Alt}_A(L, C) \longrightarrow \text{Alt}_A(L, C)$$

for the differential induced by the differential on C , and write

$$d^1: \text{Alt}_A(L, C) \longrightarrow \text{Alt}_A(L, C)$$

for the corresponding operator (1.3), interpreted suitably with respect to the grading; in other words, for a *homogeneous* multilinear alternating form f on L with values in C of degree $(n-1)$, i. e. a sequence $f = \{f_{n-1}, f_n, \dots, f_{n+\ell}, \dots\}$ of multilinear, alternating forms $f_{n+\ell}$ in $n+\ell$ variables on L with values in $C_{\ell+1}$, where $\ell \geq -1$, define d^1 by

$$(1.3') \quad \begin{aligned} (d^1 f)(\alpha_1, \dots, \alpha_n) = & (-1)^n \sum_{i=1}^n (-1)^{(i-1)} \alpha_i (f(\alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_n)) \\ & + (-1)^n \sum_{j < k} (-1)^{(j+k)} f([\alpha_j, \alpha_k], \alpha_1, \dots, \widehat{\alpha}_j, \dots, \widehat{\alpha}_k, \dots, \alpha_n). \end{aligned}$$

Then d^1 endows $\text{Alt}_A(L, C)$ with the structure of a chain complex, and it is a standard fact that so does d^0 . Moreover, since the action of L on C is assumed compatible with the differential on C , the operator d^1 is also compatible with the differential d^0 since the latter is induced from the differential on C , and this means that

$$(1.6.2.1) \quad 0 = d^0 d^1 + d^1 d^0;$$

it is precisely at this stage where the sign $(-1)^n$ in (1.3') is needed. Consequently

$$(1.6.2.2) \quad d = d^0 + d^1$$

endows $\text{Alt}_A(L, C)$ with the structure of a chain complex over R .

Notice that in the special case where C is concentrated in degree zero, that is, where C is just an A -module, $Z^0(\text{End}(C)) = \text{End}(C)$, and the differential d boils down to the usual Lie-algebra cohomology differential (1.3).

1.6.3. DIFFERENTIAL GRADED ALGEBRAS OVER A . We shall refer to a differential graded algebra E over A as a *differential graded (A, L) -algebra* if the structure maps $m: E \otimes_A E \rightarrow E$ and $\eta: A \rightarrow E$ are morphisms of (A, L) -chain complexes. Equivalently, write $Z^0(\text{Der}(E))$ for the homogeneous derivations of E of degree zero that are also chain maps; then E is a differential graded (A, L) -algebra if and only if the structure map ω factors through an action $\omega: L \rightarrow Z^0(\text{Der}(E))$ (denoted still by ω with a slight abuse of notation). The same kind of reasoning as above shows that, for a differential graded (A, L) -algebra E , the pairing (1.5') and the differential (1.6.2.2) endow the A -multilinear forms $\text{Alt}_A(L, E)$ with values in E with the structure of a differential graded algebra over R . In fact, in view of what was said above, the operator d^1 endows $\text{Alt}_A(L, E)$ with the structure of a differential graded algebra, and it is well known that so does d^0 and, by virtue of (1.6.2.1), the operator (1.6.2.2) is a differential, too.

Notice in the special case where E is concentrated in degree zero, that is, where E is just an algebra over A , $Z^0(\text{Der}(E)) = \text{Der}(E)$, and the differential d boils down to the usual Lie algebra cohomology differential (1.3).

1.6.4. DIFFERENTIAL GRADED COALGEBRAS OVER A . Recall that a *differential graded coalgebra* C over A is a chain complex C over A together with chain maps $\Delta: C \rightarrow C \otimes_A C$ and $\varepsilon: C \rightarrow A$ over A that satisfy the usual properties. Recall also that a *coderivation* $\phi: C \rightarrow C$ of a differential graded coalgebra C is a morphism of the underlying graded R -modules so that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\phi} & C \\ \Delta \downarrow & & \Delta \downarrow \\ C \otimes_A C & \xrightarrow{\phi \otimes_A \text{Id} + \text{Id} \otimes_A \phi} & C \otimes_A C \end{array}$$

is commutative. Notice that the differential of C is itself a coderivation. We shall refer to a differential graded coalgebra C over A as a *differential graded (A, L) -coalgebra* if the structure maps $\Delta: C \rightarrow C \otimes_A C$ and $\varepsilon: C \rightarrow A$ are morphisms of (A, L) -chain complexes. Equivalently, write $Z^0(\text{Coder}(C))$ for the homogeneous coderivations of C of degree zero that are also chain maps; then C is a differential graded (A, L) -coalgebra if and only if the structure map ω factors through an action $\omega: L \rightarrow Z^0(\text{Coder}(C))$ (denoted still by ω , with a slight abuse of notation).

For later reference we reproduce the notions of cofree graded coalgebra and that of cofree graded cocommutative coalgebra. Let Y be a graded A -module. Consider the graded tensor coalgebra $(T'_A[Y], \Delta)$ over A on Y . It may be written

$$(1.6.4.1) \quad T'_A[Y] = \sum \oplus T_A^{(n)}[Y],$$

where $T_A^{(n)}[Y] = Y^{\otimes_A n}$, and where, for $0 \leq k \leq n$, the component

$$(1.6.4.2) \quad T_A^{(n)}[Y] \longrightarrow T_A^{(k)}[Y] \otimes_A T_A^{(n-k)}[Y]$$

of the diagonal map Δ is the obvious isomorphism; we note that the direct sum and the tensor product are here understood in the graded sense. For later reference, we spell out the following.

Finiteness hypothesis 1.6.4.3.f. *The graded A -module Y is concentrated in non-negative degrees and finitely generated or has the property that Y_0 is finitely generated and Y_i is non-zero only in finitely many degrees $i > 0$.*

Let $\pi: T'_A[Y] \rightarrow Y$ be the canonical projection. The tensor coalgebra satisfies the following universal property:

1.6.4.3. *Suppose that Y satisfies the finiteness hypothesis (1.6.4.3.f). Then given any graded coalgebra C over A and a morphism $\phi: C \rightarrow Y$ of graded A -modules, there is a unique morphism $\Phi: C \rightarrow T'_A[Y]$ of graded coalgebras over A so that $\pi\Phi = \phi$.*

Indeed, given $\phi: C \rightarrow Y$, let $\Phi = \sum \phi_i$, where $\phi_0 = \varepsilon$ and where, for $i \geq 1$, ϕ_i denotes the composite

$$\phi_i: C \xrightarrow{\Delta^{(i)}} C^{\otimes_A i} \xrightarrow{\phi} Y^{\otimes_A i}.$$

Here $\Delta^{(i)}$ refers to some corresponding iterate of the diagonal map; it does not matter which one we take, by coassociativity. In view of the finiteness hypothesis

(1.6.4.3.f), the map Φ is well defined, that is, in each degree, only finitely many terms ϕ_i are non-zero.

In other words, under the finiteness hypothesis (1.6.4.3.f), the triple $(T'_A[Y], \Delta, \pi)$ constitutes the *cofree graded coalgebra* on Y over A . We note that, in case Y does not satisfy a finiteness hypothesis of the kind (1.6.4.3.f), the tensor coalgebra on Y will not satisfy the universal property; it must then be completed, and so must be the tensor product which is the target for the diagonal map. In the application in Section 3 this problem will not occur. Moreover, the obvious morphism $\eta: A \rightarrow T'_A[Y]$ of coalgebras over A endows the tensor coalgebra with the structure of a *coaugmentation*. Consequently the tensor coalgebra is filtered by the *coaugmentation filtration* (see e. g. [26]). This filtration of $T'_A[Y]$ is the same as that by the length of tensors in each summand $T_A^{(n)}[Y]$.

Recall that a graded coalgebra C over A is *cocommutative* provided the composite of the diagonal Δ with the interchange map

$$C \otimes_A C \longrightarrow C \otimes_A C, \quad a \otimes_A b \mapsto (-1)^{|a||b|} b \otimes_A a,$$

coincides with Δ . Recall that the *cofree graded cocommutative coalgebra on* (the graded A -module) Y *in the category of A -modules* is a pair $(S'_A[Y], \pi)$, where $S'_A[Y]$ is a graded cocommutative coalgebra over A and where $\pi: S'_A[Y] \rightarrow Y$ is a morphism of A -modules having the following universal property:

1.6.4.3.c. *Given any graded cocommutative coalgebra C over A and a morphism $\phi: C \rightarrow Y$ of A -modules, there is a unique morphism $\Phi: C \rightarrow S'_A[Y]$ of graded (commutative) coalgebras over A so that $\pi\Phi = \phi$.*

Instead of “cofree graded cocommutative coalgebra on Y ” we shall also say *graded symmetric coalgebra on Y* (over A). It is clear that the graded symmetric coalgebra on Y is unique up to isomorphism, if it exists.

We now reproduce a construction of the graded symmetric coalgebra on Y : Let $(T'_A[Y], \Delta)$ be the tensor coalgebra over A on Y . For $n \geq 1$, let $T_A^{(n)}[Y]$ be its homogeneous degree n component, let the symmetric group S_n on n letters act on $T_A^{(n)}[Y]$ in the graded sense in the obvious way, that is to say, for any $y = y_1 \otimes_A \cdots \otimes_A y_n \in T_A^{(n)}[Y]$ and for a transposition $\tau = (i, j)$ avec $i < j$, we have

$$\tau(y_1 \otimes_A \cdots \otimes_A y_i \otimes_A \cdots \otimes_A y_j \otimes_A \cdots \otimes_A y_n) = \varepsilon(i, j, y)(y_1 \otimes_A \cdots \otimes_A y_j \otimes_A \cdots \otimes_A y_i \otimes_A \cdots \otimes_A y_n)$$

where

$$\varepsilon(i, j, y) = (-1)^{|y_i||y_j| + (|y_{i+1}| + \cdots + |y_{j-1}|)(|y_i| + |y_j|)},$$

and let $(S'_A)^n[Y]$ be the submodule of S_n -invariants. Let $S'_A[Y] = \oplus (S'_A)^n[Y]$, and let $\pi: S'_A[Y] \rightarrow Y$ be the obvious projection. For any degree n , given $0 \leq k \leq n$, the component (1.6.4.2) of the restriction of the diagonal map Δ to $T_A^{(n)}[Y]$ then maps an element $x \in (S'_A)^n[Y]$, that is, an element $x \in T_A^{(n)}[Y]$ invariant under S_n , to an element in $T_A^{(k)}[Y] \otimes_A T_A^{(n-k)}[Y]$ invariant under $S_k \times S_{n-k}$, with respect to the obvious action of the latter group on $T_A^{(k)}[Y] \otimes_A T_A^{(n-k)}[Y]$; indeed, since by construction, the morphism (1.6.4.2) is just the obvious isomorphism, this amounts to the fact that

$x \in T_A^{(n)}[Y]$ is invariant under $S_k \times S_{n-k}$, with respect to the corresponding obvious embedding of $S_k \times S_{n-k}$ into S_n . Consequently the diagonal map Δ on $T_A'[Y]$ passes to one on $S_A'[Y]$ which we still denote by Δ . Moreover, the coaugmentation map η for $T_A'[Y]$ yields a coaugmentation map $\eta: A \rightarrow S_A'[Y]$ for $S_A'[Y]$.

Proposition 1.6.4.4. *Suppose that Y satisfies the finiteness hypothesis (1.6.4.3.f). Then $(S_A'[Y], \Delta, \pi)$ is the graded symmetric coalgebra on Y over A .*

Proof. It is clear that, under the finiteness hypothesis (1.6.4.3.f), $(S_A'[Y], \Delta, \pi)$ satisfies the universal property (1.6.4.3.c). \square

In view of what was said above, the coaugmentation filtration of $S_A'[Y]$ is the same as that by the length of invariant tensors in $T_A^{(n)}[Y]$.

Proposition 1.6.4.5. *For a graded (A, L) -module Y , the usual formula*

$$\alpha(y_1 \otimes_A y_2 \otimes_A \cdots \otimes_A y_n) = \sum y_1 \otimes_A y_2 \otimes_A \cdots \otimes_A \alpha(y_i) \otimes_A \cdots \otimes_A y_n, \quad \alpha \in L,$$

endows

- (1) *the graded tensor coalgebra $(T_A'[Y], \Delta)$ with a structure of a graded (A, L) -coalgebra, and*
- (2) *the graded tensor algebra $(T_A[Y], m)$ with a structure of a graded (A, L) -algebra.*

Furthermore, if Y satisfies the finiteness hypothesis (1.6.4.3.f) so that the graded symmetric coalgebra $(S_A'[Y], \Delta)$ exists, the latter inherits a structure of a graded (A, L) -coalgebra.

Proof. This is left to the reader. \square

When Y is concentrated in odd degrees, $S_A'[Y]$ is just the (graded) exterior coalgebra $\Lambda_A'[Y]$; when Y is, furthermore, projective as a graded A -module, as graded A -modules, the exterior algebra $\Lambda_A[Y]$ and coalgebra $\Lambda_A'[Y]$ coincide; the structures in fact combine to that of a graded Hopf algebra. More precisely, the usual diagonal map

$$(1.6.4.6) \quad \Delta: \Lambda_A[Y] \longrightarrow \Lambda_A[Y] \otimes_A \Lambda_A[Y]$$

determined by

$$(1.6.4.7) \quad \Delta(v) = v \otimes_A 1 + 1 \otimes_A v, \quad v \in Y,$$

endows the graded exterior algebra $\Lambda_A[Y]$ with the structure of a graded commutative and graded cocommutative Hopf algebra and hence in particular with that of a graded cocommutative coalgebra; see e. g. MAC LANE [20] for details. The latter is precisely the graded exterior A -coalgebra structure on Y . Since the property of being a Hopf algebra implies in particular that its diagonal map is multiplicative, the rule (1.6.4.7) in fact completely determines (1.6.4.6). Explicitly, given $x_1, \dots, x_{p+q} \in Y$, the value

$$\Delta(x_1 \wedge_A \cdots \wedge_A x_{p+q}) \in \Lambda_A[Y]$$

is given by the formula

$$(1.6.4.8) \quad \begin{aligned} & \Delta(x_1 \wedge_A \cdots \wedge_A x_{p+q}) \\ &= \sum_{\sigma} \text{sign}(\sigma) (x_{\sigma(1)} \wedge_A \cdots \wedge_A x_{\sigma(p)}) \otimes_A (x_{\sigma(p+1)} \wedge_A \cdots \wedge_A x_{\sigma(p+q)}), \end{aligned}$$

where σ runs through (p, q) -shuffles and where $\text{sign}(\sigma)$ refers to the sign of σ .

Likewise, when Y is concentrated in even degrees, $S'_A[Y]$ is the (*graded*) *symmetric coalgebra* $\Sigma'_A[Y]$ in the category of A -modules, but the relationship between the graded symmetric algebra $\Sigma_A[Y]$ and the graded symmetric coalgebra cannot in general be explained in terms of an isomorphism of Hopf algebras. This relationship is actually folk-lore but difficult to locate in the literature. We therefore spell out some of the details.

We still suppose that Y is projective as a graded A -module. The diagonal map

$$\Delta: Y \rightarrow Y \oplus Y, \quad \Delta(y) = (y, y), \quad y \in Y,$$

induces a diagonal map

$$(1.6.4.9) \quad \Delta: \Sigma_A[Y] \rightarrow \Sigma_A[Y] \otimes_A \Sigma_A[Y] \cong \Sigma_A[Y \oplus Y]$$

which endows $\Sigma_A[Y]$ with the structure of a (graded) commutative and cocommutative Hopf algebra. Explicitly, given $x_1, \dots, x_{p+q} \in Y$, the value

$$\Delta(x_1 x_2 \cdots x_{p+q}) \in \Sigma_A[Y]$$

is given by the formula

$$(1.6.4.10) \quad \Delta(x_1 x_2 \cdots x_{p+q}) = \sum_{\sigma} (x_{\sigma(1)} \cdots x_{\sigma(p)}) \otimes_A (x_{\sigma(p+1)} \cdots x_{\sigma(p+q)})$$

where σ runs through (p, q) -shuffles. This diagonal map is referred to as *shuffle coproduct*. In view of the universal property (1.6.4.3.c) of the symmetric coalgebra $\Sigma'_A[Y]$, the canonical projection $\phi: \Sigma_A[Y] \rightarrow Y$ induces a morphism

$$(1.6.4.11) \quad \Sigma_A[Y] \rightarrow \Sigma'_A[Y]$$

of graded (commutative) coalgebras. Furthermore, again in view of the universal property (1.6.4.3.c) of the symmetric coalgebra $\Sigma'_A[Y]$, addition

$$Y \oplus Y \rightarrow Y, \quad (y_1, y_2) \mapsto y_1 + y_2, \quad y_1, y_2 \in Y,$$

induces a multiplication map

$$(1.6.4.12) \quad \Sigma'_A[Y] \otimes_A \Sigma'_A[Y] \cong \Sigma'_A[Y \oplus Y] \rightarrow \Sigma'_A[Y]$$

which endows $\Sigma'_A[Y]$ with the structure of a (graded) commutative and cocommutative Hopf algebra as well, and (1.6.4.11) is a morphism of Hopf algebras. Since $\Sigma_A[Y]$ is the free graded commutative algebra on Y , the morphism of graded algebras which

underlies (1.6.4.11) may also be obtained as the unique morphism of graded algebras induced by the canonical inclusion from Y into $\Sigma'_A[Y]$, viewed as a graded algebra.

When we dualize $\Sigma'_A[Y]$ and $\Sigma_A[Y]$, we obtain the graded Hopf algebras

$$\Sigma_A[Y^*] = \text{Hom}_A(\Sigma'_A[Y], A)$$

and

$$\Sigma_A^*[Y] = \text{Hom}_A(\Sigma_A[Y], A) = \Sigma'_A[Y^*]$$

respectively, where $Y^* = \text{Hom}_A(Y, A)$; here $\Sigma_A[Y^*]$ is the algebra of *graded polynomial functions* on Y and $\Sigma_A^*[Y]$ that of *graded symmetric functions* on Y ; the multiplication on the algebra of graded symmetric functions is the dual of the shuffle coproduct (1.6.4.9) and hence the usual *shuffle product*. The dual of (1.6.4.11) is the canonical map

$$(1.6.4.13) \quad \Sigma_A[Y^*] \rightarrow \Sigma_A^*[Y]$$

from the Hopf algebra of graded polynomial functions to that of graded symmetric functions on Y . It sends a polynomial function to the corresponding symmetric function and is formally exactly of the same kind as (1.6.4.11), with Y^* instead of Y .

When Y is A -free, after a choice of basis $\{e_1, e_2, \dots\}$ of Y has been made and when $\{\xi_1, \xi_2, \dots\}$ refers to the dual basis of Y^* , $\Sigma_A[Y^*]$ is the polynomial A -Hopf algebra $A[\xi_1, \xi_2, \dots]$ on ξ_1, ξ_2, \dots whereas $\Sigma_A^*[Y]$ is the *divided polynomial A -Hopf algebra* $\Gamma[\xi_1, \xi_2, \dots]$ on ξ_1, ξ_2, \dots ,

that is, $\Sigma_A^*[Y]$ is the graded commutative A -Hopf algebra generated by $\gamma_k \xi_j$, $k, j \geq 1$, subject to the relations

$$k! \gamma_k \xi_j = \xi_j^k, \quad k, j \geq 1,$$

with A -coalgebra structure Δ determined by

$$\Delta \gamma_k \xi_j = \sum_{u+v=k} \gamma_u \xi_j \otimes_A \gamma_v \xi_j.$$

See [4] for more details on divided powers. The map (1.6.4.13) is then the obvious one which sends the multiplicative generator ξ_j to the multiplicative generator $\gamma_1 \xi_j = \xi_j$ but (1.6.4.13) is not in general an isomorphism. In characteristic zero it is an isomorphism, though, since we can then define the divided power operations in $A[\xi_1, \xi_2, \dots]$ by

$$\gamma_k \xi_j = \frac{1}{k!} \xi_j^k, \quad j, k \geq 1;$$

the inverse mapping of (1.6.4.13) is then usually referred to as *polarization*.

1.6.5. CUP PRODUCTS. Let C be a differential graded coalgebra in the category of A -modules, with structure map $C \xrightarrow{\Delta} C \otimes_A C$, and let

$$(1.6.5.1) \quad \mu_A: M' \otimes_A M'' \longrightarrow M$$

be a pairing of differential graded A -modules, that is, of chain complexes in the category of A -modules. Given morphisms $a: C \rightarrow M'$ and $b: C \rightarrow M''$ (say) of the underlying graded modules, the *cup product* of a and b with respect to μ_A , written $a \cup b$, is the composite

$$(1.6.5.2) \quad C \xrightarrow{\Delta} C \otimes_A C \xrightarrow{a \otimes_A b} M' \otimes_A M'' \xrightarrow{\mu_A} M;$$

see [14, 15, 24, 26]. Here the tensor product $a \otimes_A b$ is the *graded one*, that is to say,

$$(1.6.5.3) \quad a \otimes_A b(x \otimes_A y) = (-1)^{|b||x|} \mu_A(a(x) \otimes_A b(y)), \quad x \in M', \quad y \in M''.$$

The resulting pairing

$$(1.6.5.4) \quad \cup: \text{Hom}_A(C, M') \otimes_A \text{Hom}_A(C, M'') \rightarrow \text{Hom}_A(C, M), \quad (a, b) \mapsto a \cup b,$$

is the usual *cup pairing* with respect to μ_A ; it is associative in the obvious way.

To relate this kind of cup pairing with the pairing (1.5''), given an (R, A) -Lie algebra L , let $Y = sL$, the *suspension* sL of L ; for the present purposes this means that sL is just L except that its elements are regraded up by one. Given a pairing of differential graded A -modules of the kind (1.6.5.1), we then have the corresponding pairing (1.6.5.4). In the special case where the pairing (1.6.5.1) is one of ungraded A -modules, viewed as differential graded A -modules concentrated in degree zero, the resulting pairing (1.6.5.4) is exactly the same as the pairing (1.5'') above. It is exactly at this stage where the sign $(-1)^{|\alpha||\beta|}$ in the formula (1.4.2) above is needed.

In the general (graded) case, for an arbitrary differential graded coalgebra C and a differential graded algebra U , both in the category of A -modules, the cup product turns $\text{Hom}_A(C, U)$ into a differential graded algebra in the category of A -modules, with unit $\eta\varepsilon$ and differential D given by

$$(1.6.5.5) \quad Df = df + (-1)^{(|f|+1)} fd,$$

for homogeneous $f: C \rightarrow U$; further, if C and U have a coaugmentation map η and augmentation map ε , respectively, the assignment $\varphi \mapsto \varepsilon\varphi\eta$ yields an augmentation map for this differential graded algebra.

2. Extensions

The algebraic analog of an “Atiyah sequence” or of a “transitive Lie algebroid”, see e. g. [21] or (2.2) below, is an extension of Lie-Rinehart algebras. In the present section we study such extensions by means of generalizations of the usual notions of connection and curvature in a principal bundle.

Let L', L, L'' be (R, A) -Lie algebras. An *extension* of (R, A) -Lie algebras is a short exact sequence

$$(2.1) \quad \mathbf{e}: 0 \rightarrow L' \rightarrow L \xrightarrow{p} L'' \rightarrow 0$$

in the category of (R, A) -Lie algebras; notice in particular that the Lie algebra L' necessarily acts trivially on A . If also $\bar{\mathbf{e}}: 0 \rightarrow L' \rightarrow \bar{L} \rightarrow L'' \rightarrow 0$ is an extension of (R, A) -Lie algebras, as usual, \mathbf{e} and $\bar{\mathbf{e}}$ are said to be *congruent*, if there is a morphism $(\text{Id}, \cdot, \text{Id}): \mathbf{e} \rightarrow \bar{\mathbf{e}}$ of extensions of (R, A) -Lie algebras.

REMARK 2.2. Let N be a smooth finite dimensional manifold, let A be the algebra of smooth functions on N , and let $\xi: P \rightarrow N$ be a principal bundle, with structure group G acting from the right. The vertical subbundle $\psi: V \rightarrow P$ of the tangent bundle τ_P of P is well known to be trivial, having as fibre the Lie algebra \mathfrak{g} of G , that is, $V \cong P \times \mathfrak{g}$. Dividing out the actions of G from the right, we obtain an extension

$$(2.2.1) \quad 0 \rightarrow \text{ad}(\xi) \rightarrow \tau_P/G \rightarrow \tau_N \rightarrow 0$$

of vector bundles over N , where τ_N is the tangent bundle of N . This sequence has been introduced by ATIYAH [2] (Theorem 1) and is now usually called the *Atiyah sequence* of the principal bundle ξ ; here $\text{ad}(\xi)$ is the bundle associated to the principal bundle by the adjoint representation of G on its Lie algebra \mathfrak{g} . A complete account to Atiyah sequences may be found in App. A of [21]. The spaces $\mathfrak{g}(\xi) = \Gamma \text{ad}(\xi)$ and $E(\xi) = \Gamma(\tau_P/G)$ of sections inherit obvious structures of Lie algebras, in fact of (\mathbb{R}, A) -Lie algebras, and

$$(2.2.2) \quad 0 \rightarrow \mathfrak{g}(\xi) \rightarrow E(\xi) \rightarrow \text{Vect}(N) \rightarrow 0$$

is an extension of (\mathbb{R}, A) -Lie algebras; here $\text{Vect}(N) = \Gamma(\tau_N)$, the Lie algebra of vector fields on N , and $\mathfrak{g}(\xi)$ is in an obvious way the Lie algebra of the *group of gauge transformations* of ξ .

We now generalize the classical notions of principal connection and curvature: Let \mathbf{e} be an extension (2.1) of (R, A) -Lie algebras, and suppose that it splits in the category of A -modules; this will e. g. hold if L'' is projective as an A -module. Then \mathbf{e} may be represented by a 2-cocycle: Let $\omega: L'' \rightarrow L$ be a section of A -modules for the projection $p: L \rightarrow L''$. We refer to ω as an *e-connection*. Given an *e-connection*, define the corresponding (*e-connection*) *curvature* $\Omega: L'' \otimes_A L'' \rightarrow L'$ as the morphism Ω of A -modules satisfying

$$(2.3) \quad [\omega(\alpha), \omega(\beta)] = \omega[\alpha, \beta] + \Omega(\alpha, \beta)$$

for every $\alpha, \beta \in L''$. The usual reasoning reveals that Ω is indeed well defined as an alternating A -bilinear 2-form on L'' with values in L' ; under the circumstances

of (2.2), this amounts to Ω being a tensor. These notions of **e**-connection and **e**-curvature generalize the concepts of principal connection and principal curvature; indeed, under the circumstances of (2.2), they come down to their descriptions in the language of Atiyah sequences due to MACKENZIE [21].

In [11] we have shown that in view of the Jacobi identity in L the morphism Ω must satisfy a 2-cocycle condition phrased in terms of a suitable notion of covariant derivative which generalizes the usual Bianchi identity. We now explain this somewhat more formally:

The adjoint representation $\text{ad}: L \rightarrow \text{End}(L')$ endows the A -Lie algebra L' with a structure of an (A, L) -module, in fact with that of an (A, L) -Lie algebra

$$(2.4) \quad \text{ad}: L \longrightarrow \text{Der}(L')$$

in the sense of (1.6.1). In particular, we have the chain complex $\text{Alt}_A(L, L')$ with the differential d_L given by (1.3). Furthermore, the projection $p: L \rightarrow L''$ induces an injection

$$p^*: \text{Alt}_A(L'', L') \longrightarrow \text{Alt}_A(L, L')$$

of graded A -modules, and the chosen **e**-connection ω induces a surjection

$$\omega^*: \text{Alt}_A(L, L') \longrightarrow \text{Alt}_A(L'', L')$$

of graded A -modules. Define the operator D^ω of *covariant derivative* on the graded A -module $\text{Alt}_A(L'', L')$ as the composite

$$(2.5.1) \quad D^\omega = \omega^* d_L p^*: \text{Alt}_A(L'', L') \longrightarrow \text{Alt}_A(L'', L').$$

When we write out this operator, we obtain the usual formula

$$(2.5.2) \quad \begin{aligned} (D^\omega f)(\alpha_1, \dots, \alpha_n) &= (-1)^n \sum_{i=1}^n (-1)^{(i-1)} \text{ad}(\omega(\alpha_i))(f(\alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_n)) \\ &\quad + (-1)^n \sum_{j < k} (-1)^{(j+k)} f([\alpha_j, \alpha_k], \alpha_1, \dots, \widehat{\alpha}_j, \dots, \widehat{\alpha}_k, \dots, \alpha_n). \end{aligned}$$

It is readily seen that the Jacobi identity in L boils down to the identity

$$(2.5.3) \quad D^\omega(\Omega) = 0$$

which generalizes the Bianchi identity; we refer to (2.5.3) as the *generalized Bianchi identity*.

The 2-cocycle Ω is uniquely determined by **e** up to a coboundary; see Section 2 in [11] for details. Moreover, from that paper, we recall the following.

Theorem 2.6. *Given an (R, A) -Lie algebra L'' and an (A, L'') -module L' , viewed as an abelian (A, L'') -Lie algebra, the assignment, to an extension which splits in the category of A -modules and realizes the (A, L'') -module structure on L' , of its 2-cocycle $\Omega \in \text{Alt}_A(L'', L')$, yields a bijective correspondence between the congruence classes of such extensions of L' by L'' and the elements of $H^2(\text{Alt}_A(L'', L'))$.*

When L' is non-abelian, the generalized Bianchi identity (2.5.3) says that the 2-form Ω is a non-abelian 2-cocycle, and hence Ω does *not* lead to a cohomology class in a naive way. In this case, the classical argument due to EILENBERG-MAC LANE [8], see e. g. (IV.8.8) in MAC LANE [20], suitably rephrased for the present case, shows that the cohomology group $H_A^2(\text{Alt}_A(L'', Z))$ acts faithfully and transitively on the congruence classes of extensions of L'' by L' with the same “outer action” of L'' on L' where Z refers to the center of L' as an A -Lie algebra. It seems worthwhile giving some of the details: At first, the term “outer action” means the following: The Lie algebra L' acts on itself by means of the adjoint representation $\text{ad}: L' \rightarrow \text{Der}(L')$ and the image is well known to be a Lie ideal; hence the morphism ad admits a cokernel, the Lie algebra $\text{ODer}(L')$ of *outer* derivations of L' . We refer to an arbitrary morphism $L'' \rightarrow \text{ODer}(L')$ of R -Lie algebras as an *outer action* of L'' on L' . An outer action induces an action

$$L'' \rightarrow \text{Der}(Z)$$

that endows the center Z of L' with a structure of an (A, L'') -module, in fact, with that of an (A, L'') -Lie algebra in the sense of (1.6.1), with trivial Lie structure on Z understood. In particular, the chain complex $\text{Alt}_A(L'', Z)$ is well defined. Furthermore, given an extension \mathbf{e} of Lie-Rinehart algebras of the kind (2.1), the corresponding action (2.4) induces an outer action

$$(2.7.1) \quad L'' \rightarrow \text{ODer}(L')$$

of L'' on L' . Next, addition induces an operation

$$+: L' \oplus Z \longrightarrow L'$$

of A -modules which, in turn, induces an operation

$$(2.7.2) \quad +: \text{Alt}_A(L'', L') \oplus \text{Alt}_A(L'', Z) \longrightarrow \text{Alt}_A(L'', Z).$$

A choice of \mathbf{e} -connection corresponds to a decomposition of L as a direct sum $L' \oplus L''$ as A -modules, and the aforementioned argument due to EILENBERG-MAC LANE [8], suitably rephrased, establishes a proof of the following.

Theorem 2.7. *Given an extension \mathbf{e} of Lie-Rinehart algebras of the kind (2.1), the operation (2.7.2) induces a faithful and transitive action of the cohomology group $H^2(\text{Alt}_A(L'', Z))$ on the congruence classes of extensions of L'' by L' with “outer action” (2.7.1) of L'' on L' in such a way that, when Ω is the curvature corresponding to an extension \mathbf{e} and when $\rho \in \text{Alt}_A^2(L'', Z)$ is a 2-cocycle, the 2-form*

$$\Omega + \rho$$

is the curvature corresponding to an extension \mathbf{e}_ρ representing the congruence class obtained by operation upon the class of \mathbf{e} with $[\rho] \in H^2(\text{Alt}_A(L'', Z))$. \square

A version of this result in the framework of transitive Lie algebroids may be found in (IV.3.31) of [21]. An \mathbf{e} -connection is said to be *flat* if its curvature is zero. It is clear that an extension \mathbf{e} admits a flat \mathbf{e} -connection if and only if it splits in the category of (R, A) -Lie algebras.

In the next section we shall need a suitable notion of an operator of covariant derivative for (A, L) -modules. We now explain this briefly: Let M be an (A, L) -module, possibly graded. Then the restriction to L' of the structure map from L to $\text{End}_R(M)$ is an action ϕ of L' on M in the usual sense of Lie algebra actions in the category of A -modules. Consider the chain complex $\text{Alt}_A(L, M)$, with the Lie algebra cohomology differential d_L , cf. (1.3). The projection $p: L \rightarrow L''$ induces an injection

$$p^*: \text{Alt}_A(L'', M) \longrightarrow \text{Alt}_A(L, M)$$

of graded A -modules. Pick an \mathbf{e} -connection ω ; this induces a surjection

$$\omega^*: \text{Alt}_A(L, M) \longrightarrow \text{Alt}_A(L'', M)$$

of graded A -modules and hence an operator D^ω of *covariant derivative* on the graded A -module $\text{Alt}_A(L'', M)$, given as the composite

$$(2.8.1) \quad D^\omega = \omega^* d_L p^*: \text{Alt}_A(L'', M) \longrightarrow \text{Alt}_A(L'', M).$$

A version of this may be found in (IV.3.9) of [21]. When we write out (2.8.1) we obtain the usual formula

$$(2.8.2) \quad \begin{aligned} & (D^\omega f)(\alpha_1, \dots, \alpha_n) \\ &= (-1)^n \sum_{i=1}^n (-1)^{(i-1)} (\omega(\alpha_i)) (f(\alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_n)) \\ &+ (-1)^n \sum_{j < k} (-1)^{(j+k)} f([\alpha_j, \alpha_k], \alpha_1, \dots, \widehat{\alpha}_j, \dots, \widehat{\alpha}_k, \dots, \alpha_n). \end{aligned}$$

It is manifest that the \mathbf{e} -curvature $\Omega: L'' \otimes_A L'' \rightarrow L'$ of ω , combined with the Lie algebra action $\phi: L' \rightarrow \text{End}_A(M)$ of L' on M in the category of A -modules mentioned earlier is then the adjoint of

$$(2.9) \quad D^\omega D^\omega: M \longrightarrow \text{Alt}_A^2(L'', M).$$

Explicitly, with $D = D^\omega$, this is formally the usual formula

$$\phi \Omega(\alpha, \beta) = D_\alpha D_\beta - D_\beta D_\alpha - D_{[\alpha, \beta]}$$

where

$$D_\alpha(m) = (\omega(\alpha))(m), \quad \alpha \in L, \quad m \in M.$$

For a general (R, A) -Lie algebra L and an A -module M , there are notions of L -connection and L -curvature, and these can be realized by an action on M of a suitable extension E of L by $\text{End}_A(M)$ in the category of (R, A) -Lie algebras. We do not need the details here; see e. g. [11] (Section 2).

3. The Chern-Weil construction

As before we suppose that the extension (2.1) splits in the category of A -modules. Let s^2L' be the double suspension of L' , i. e. s^2L' is just L' , except that its elements are regraded up by 2. The graded algebra $\text{Hom}_A(\Sigma'_A[s^2L'], A)^L$ being equipped with the zero differential, we shall construct a morphism

$$\text{Hom}_A(\Sigma'_A[s^2L'], A)^L \longrightarrow \text{Alt}_A(L'', A)$$

of differential graded commutative R -algebras whose induced morphism on homology depends only on the congruence class (cf. Section 2) of the extension (2.1). When L' is finitely generated and projective, the graded A -algebra $\text{Hom}_A(\Sigma'_A[s^2(L')], A)$ may be identified with the symmetric A -algebra $\Sigma_A[s^2(L')^*]$ on the A -dual $s^2(L')^*$ of $s^2(L')$ as indicated. In particular, when L' is A -free of finite type, $\text{Hom}_A(\Sigma'_A[s^2(L')], A)$ is the polynomial algebra over A on an A -basis of $\text{Hom}_A(s^2(L'), A)$.

By assumption, L' is a Lie algebra over A in the usual sense. Furthermore, the extension (2.1) splits in the category of A -modules; let $\omega: L'' \rightarrow L$ be an \mathbf{e} -connection for (2.1), and let

$$(3.1) \quad \Omega: L'' \otimes_A L'' \longrightarrow L'$$

be its curvature. Since Ω is an A -bilinear alternating form, it passes through the second exterior power $\Lambda_A^2[L'']$ and hence induces a morphism

$$\Lambda_A^2 \Omega: \Lambda_A^2[L''] \longrightarrow L'$$

of A -modules. Thus, in view of (1.6.4.3.c), $\Lambda'_A[sL'']$ being endowed with the graded exterior A -coalgebra structure (1.6.4.6), the curvature Ω induces a morphism

$$(3.2) \quad \Omega_{\sharp}: \Lambda'_A[sL''] \longrightarrow \Sigma'_A[s^2L']$$

of graded coalgebras over A in the following way: Let $\pi: \Sigma'_A[s^2L'] \rightarrow s^2L'$ be the projection map which is part of the structure of the graded symmetric coalgebra over A (cf. (1.6.4)), and let

$$\Omega_{\natural}: \Lambda_A^2[sL''] \rightarrow s^2L', \quad \Omega_{\flat}: \Lambda'_A[sL''] \longrightarrow s^2L'$$

be the homogeneous degree zero morphisms of graded A -modules determined by the requirement that the diagram

$$(3.3) \quad \begin{array}{ccc} \Lambda'_A[sL''] & \xrightarrow{\Omega_{\flat}} & s^2L' \\ \text{proj} \downarrow & & \text{Id} \downarrow \\ \Lambda_A^2[sL''] & \xrightarrow{\Omega_{\natural}} & s^2L' \\ \Lambda_A^2 s \uparrow & & s^2 \uparrow \\ \Lambda_A^2[L''] & \xrightarrow{\Lambda_A^2 \Omega} & L' \end{array}$$

be commutative; here “proj” refers to the obvious projection map from $\Lambda'_A[sL'']$ to $\Lambda_A^2[sL'']$. We note that Ω_{\natural} is just the appropriate rewrite of $\Lambda_A^2\Omega$ in the formally appropriate graded setting. In view of the universal property (1.6.4.3.c) of the graded symmetric coalgebra, the induced morphism (3.2) is determined by the requirement that

$$(3.4) \quad \pi\Omega_{\natural} = \Omega_{\flat}.$$

We refer to Ω_{\natural} as a *classifying map for the extension* (2.1). It may be viewed as an algebraic analogue of the more usual notion of classifying map in topology and differential geometry. Since Ω is unique up to a coboundary, this classifying map is uniquely determined by (2.1) up to a non-abelian coboundary in a suitable sense. The construction of Ω_{\natural} is completely formal and does *not* require that L' satisfy any finiteness assumption; the reason is that we work with the graded symmetric coalgebra $\Sigma'_A[s^2L']$ which, apart from other formal advantages, in particular removes the existence problem for the cofree cocommutative coalgebra we would be faced with in general if we had tried an ungraded construction.

The classifying map (3.2) induces a morphism

$$(3.5) \quad \mathrm{Hom}_A(\Sigma'_A[s^2L'], A) \longrightarrow \mathrm{Alt}_A(L'', A)$$

of graded A -algebras. To manufacture a chain map from it, we observe that the adjoint representation of L on itself induces an action of L on L' that endows L' with the structure of an (A, L) -module (in fact, with that of an (A, L) -Lie algebra, cf. (1.6.1)). In view of (1.6.4.5), this L -action on L' induces an action

$$(3.6) \quad \omega_{\mathbf{e}}: L \rightarrow \mathrm{Coder}_R^0(\Sigma'_A[s^2L'])$$

which endows the latter with the structure of a graded (A, L) -coalgebra in a sense explained in (1.6.4). These structures, in turn, induce on $\mathrm{Hom}_A(\Sigma'_A[s^2L'], A)$ the structure of a graded commutative (A, L) -algebra in a sense explained in (1.6.3). As usual, for $\zeta: \Sigma'_A[s^2L'] \rightarrow A$ and $\alpha \in L$, the result of acting upon ζ with α is given by

$$(3.7) \quad \alpha(\zeta) = \alpha \circ \zeta - \zeta \circ \alpha: \Sigma'_A[s^2L'] \longrightarrow A$$

where, with an abuse of notation, the operators on $\Sigma'_A[s^2L']$ and A induced by α are denoted by α as well.

Theorem 3.8. *Given an extension \mathbf{e} of (R, A) -Lie algebras of the kind (2.1) and an \mathbf{e} -connection ω with curvature (3.1), the restriction of (3.5) to the invariants $\mathrm{Hom}_A(\Sigma'_A[s^2L'], A)^L$ goes into the cycles of $\mathrm{Alt}_A(L'', A)$. In other words, when $\mathrm{Hom}_A(\Sigma'_A[s^2L'], A)^L$ is endowed with the zero differential, the restriction of (3.5) yields a morphism*

$$(3.8.1) \quad \mathrm{Hom}_A(\Sigma'_A[s^2L'], A)^L \longrightarrow \mathrm{Alt}_A(L'', A)$$

of differential graded commutative R -algebras. Furthermore, the induced morphism

$$(3.8.2) \quad \mathrm{Hom}_A(\Sigma'_A[s^2L'], A)^L \longrightarrow \mathrm{H}^{2*}(\mathrm{Alt}_A(L'', A))$$

depends only on the congruence class of the extension (2.1) and not on a particular choice of an \mathbf{e} -connection ω etc.

We shall refer to the morphism (3.8.2) as the *Chern-Weil map* for the extension (2.1) of (R, A) -Lie algebras. The R -algebra $\text{Hom}(\Sigma'_A[s^2L'], A)^L$ will be referred to as the *algebra of characteristic classes* for the extension (2.1).

The proof of (3.8) will be subdivided into several steps. It will be convenient to view the morphism Ω_\sharp of graded coalgebras over A as an element of the graded A -module

$$(3.9) \quad \text{Hom}_A(\Lambda'_A[sL''], \Sigma'_A[s^2L']) = \text{Alt}_A(L'', \Sigma'_A[s^2L']).$$

As already pointed out, the adjoint action (3.6) induces an action of L on $\Sigma'_A[s^2L']$ that endows the latter with the structure of an (A, L) -module, in fact, with that of an (A, L) -coalgebra, and what is said at the end of Sections 2 applies, with $M = \Sigma'_A[s^2L']$. Write

$$(3.10) \quad D^\omega: \text{Alt}_A(L'', \Sigma'_A[s^2L']) \longrightarrow \text{Alt}_A(L'', \Sigma'_A[s^2L'])$$

for the corresponding operator of covariant derivative, cf. (2.8.1).

Lemma 3.11. *The classifying map Ω_\sharp satisfies*

$$(3.11.1) \quad D^\omega(\Omega_\sharp) = 0.$$

The proof requires some preparations; the proof itself will be given after (3.16) below. At first we recall from (1.6.4) that, by construction, $\Sigma'_A[s^2L'] \subseteq T'_A[s^2L']$; we denote the inclusion by ι . We recall that the graded module underlying the graded tensor coalgebra $T'_A[s^2L']$ over A coincides with that underlying the graded tensor algebra $T_A[s^2L']$ over A (but beware, the algebra and coalgebra structures are *not* compatible). The morphism ι induces an injection

$$(3.12) \quad \text{Hom}_A(\Lambda'_A[sL''], \Sigma'_A[s^2L']) \longrightarrow \text{Hom}_A(\Lambda'_A[sL''], T_A[s^2L'])$$

of graded A -modules. Moreover, in view of (1.6.5), the adjoint action of L on L' induces also an action

$$(3.13) \quad \omega_{\mathbf{e}}: L \longrightarrow \text{Der}(T_A[s^2L'])$$

of L on $T_A[s^2L']$ which endows $T_A[s^2L']$ with the structure of an (A, L) -algebra, and we can apply what is said at the end of Section 2, with $M = T_A[s^2L']$: Write

$$(3.14) \quad D^\omega: \text{Alt}_A(L'', T_A[s^2L']) \longrightarrow \text{Alt}_A(L'', T_A[s^2L'])$$

for the corresponding operator of covariant derivative. Since the values of $\omega_{\mathbf{e}}$ and hence those of the composite $\omega_{\mathbf{e}}\omega: L'' \rightarrow \text{Der}(T_A[s^2L'])$ lie in the graded R -module $\text{Der}(T_A[s^2L'])$ of derivations rather than in the full graded R -module $\text{End}(T_A[s^2L'])$ of R -linear endomorphisms, D^ω is a *derivation* over the ground ring R of the graded A -algebra

$$\text{Hom}_A(\Lambda'_A[sL''], T_A[s^2L']) = \text{Alt}_A(L'', T_A[s^2L']),$$

the A -algebra structure being the shuffle product, or cup product, cf. (1.6.4) and (1.6.5), with respect to the graded exterior A -coalgebra structure on $\Lambda'_A[sL'']$ and graded A -algebra structure on $T_A[s^2L']$. Finally, the injection (3.12) is manifestly compatible with the operations (3.10) and (3.14) of covariant derivative.

Lemma 3.15. *With respect to the graded A -coalgebra and A -algebra structures on $\Lambda'_A[sL'']$ and $T_A[s^2L']$, respectively, and with the notation Ω_b introduced in (3.3), the morphism*

$$\iota\Omega_{\sharp}: \Lambda'_A[sL''] \longrightarrow T_A[s^2L']$$

may be written

$$(3.15.1) \quad \iota\Omega_{\sharp} = \text{Id}_A + \Omega_b + (\Omega_b) \cup (\Omega_b) + (\Omega_b) \cup (\Omega_b) \cup (\Omega_b) + \dots,$$

where we do not distinguish in notation between

$$\Omega_b: \Lambda'_A[sL''] \longrightarrow s^2L'$$

and its composite with the injection $s^2L' \rightarrow T_A[s^2L']$.

In other words, we can view the morphism $\iota\Omega_{\sharp}$ as the element

$$(3.15.2) \quad \sum_{i \geq 0} (\Omega_b)^{\cup i} \in \text{Hom}_A(\Lambda'_A[sL''], T_A[s^2L'])$$

of the graded A -algebra $\text{Hom}_A(\Lambda'_A[sL''], T_A[s^2L']) = \text{Alt}_A(L'', T_A[s^2L'])$.

Proof of 3.15. This is an immediate consequence of the description of the induced morphism of coalgebras given in (1.6.4.3). \square

Corollary 3.16. *In the graded A -algebra $\text{Hom}_A(\Lambda'_A[sL''], T_A[s^2L'])$ we have*

$$D^\omega(\iota\Omega_{\sharp}) = 0.$$

Proof. From the generalized Bianchi identity (2.8.2) we know that $D^\omega(\Omega_b) = 0$. However, D^ω is a derivation over R of the graded A -algebra $\text{Hom}_A(\Lambda'_A[sL''], T_A[s^2L'])$ since, by construction, it comes from a derivation in $\text{Alt}_A(L, T_A[s^2L'])$. Hence we have, for $i \geq 2$,

$$D^\omega((\Omega_b)^{\cup i}) = \sum_{j+k=i-1} (\Omega_b)^{\cup j} \cup (D^\omega(\Omega_b)) \cup (\Omega_b)^{\cup k} = 0.$$

Hence

$$D^\omega(\iota\Omega_{\sharp}) = 0. \quad \square$$

Proof of Lemma 3.11. This follows at once from the fact that the injection (3.12) is compatible with the operations (3.10) and (3.14) of covariant derivative. \square

Proof of Theorem 3.8. It is clear that the L -action induces an L'' -action on the invariants $(\text{Hom}_A(\Sigma'_A[s^2L'], A))^{L'}$ and that the full invariants $\text{Hom}_A(\Sigma'_A[s^2L'], A)^L$ may be rewritten

$$\text{Hom}_A(\Sigma'_A[s^2L'], A)^L = (\text{Hom}_A(\Sigma'_A[s^2L'], A)^{L'})^{L''}.$$

However, since L' acts trivially on A , by adjointness, we may as well write

$$\text{Hom}_A(\Sigma'_A[s^2L'], A)^{L'} \cong \text{Hom}_A(A \otimes_{L'} \Sigma'_A[s^2L'], A)$$

where as usual $A \otimes_{L'} \Sigma'_A[s^2 L']$ refers to $\Sigma'_A[s^2 L']$ with the L' -action divided out, and the L -action on $\text{Hom}_A(\Sigma'_A[s^2 L'], A)$ passes to an L'' -action on $\text{Hom}_A(A \otimes_{L'} \Sigma'_A[s^2 L'], A)$. Furthermore, with respect to this action, we have

$$\text{Hom}_A(\Sigma'_A[s^2 L'], A)^L \cong (\text{Hom}_A(A \otimes_{L'} \Sigma'_A[s^2 L'], A))^{L''}.$$

However, since on $A \otimes_{L'} \Sigma'_A[s^2 L']$ the L' -action has been divided out, the covariant derivative (3.10) passes to the differential in $\text{Alt}_A(L'', A \otimes_{L'} \Sigma'_A[s^2 L'])$ associated to the L'' -action on $A \otimes_{L'} \Sigma'_A[s^2 L']$ and, in view of (3.11.1), the morphism $\Omega_{\#}$ passes to a cycle (say)

$$\Omega_{\#}: \Lambda'_A[s L''] \longrightarrow A \otimes_{L'} \Sigma'_A[s^2 L']$$

in $\text{Alt}_A(L'', A \otimes_{L'} \Sigma'_A[s^2 L'])$, that is to say, with respect to the differential $d = d^1$ on $\text{Alt}_A(L'', A \otimes_{L'} \Sigma'_A[s^2 L'])$ given by (1.3') (spelled out in (1.6.2)), we have

$$d(\Omega_{\#}) = 0 \in \text{Alt}_A(L'', A \otimes_{L'} \Sigma'_A[s^2 L']).$$

When we now rewrite the Chern-Weil map (3.8.2) in the form

$$\Omega_{\#}^*: \text{Hom}_A(A \otimes_{L'} \Sigma'_A[s^2 L'], A)^{L''} \longrightarrow \text{Alt}_A(L'', A)$$

we see that its image in $\text{Alt}_A(L'', A)$ consists indeed of cycles only as asserted. In fact, an element $\varphi \in \text{Hom}_A(A \otimes_{L'} \Sigma'_A[s^2 L'], A)^{L''}$ is just a morphism

$$\varphi: A \otimes_{L'} \Sigma'_A[s^2 L'] \longrightarrow A$$

of (A, L'') -modules, and $\Omega_{\#}^*(\varphi)$ coincides with the image $\varphi_*(\Omega_{\#}) \in \text{Alt}_A(L'', A)$ of the cycle $\Omega_{\#} \in \text{Alt}_A(L'', A \otimes_{L'} \Sigma'_A[s^2 L'])$ under the induced map

$$\varphi_*: \text{Alt}_A(L'', A \otimes_{L'} \Sigma'_A[s^2 L']) \longrightarrow \text{Alt}_A(L'', A),$$

that is, with the composite

$$\phi \Omega_{\#}: \Lambda'_A[s L''] \longrightarrow A.$$

Finally, a different choice $\omega': L'' \rightarrow L$ of \mathbf{e} -connection yields an \mathbf{e} -curvature Ω' so that the two cycles $\Omega_{\#}$ and $\Omega'_{\#}$ differ by a boundary. This proves Theorem 3.8. \square

REMARK 3.17. The proof given above shows in particular that the *true* global invariant for \mathbf{e} is the class

$$[\Omega_{\#}] \in H^*(\text{Alt}_A(L'', A \otimes_{L'} \Sigma'_A[s^2 L'])).$$

We finally relate our Chern-Weil construction with the classical one: Let $\xi: P \rightarrow N$ be a principal bundle with structure group a compact Lie group G , write $A = C^\infty(N)$, let $L' = \mathfrak{g}(\xi)$, viewed as an (\mathbb{R}, A) -Lie algebra with trivial action on A , and let $L = E(\xi)$ and $L'' = \text{Vect}(N)$; cf. (2.2). Consider the extension (cf. (2.2.2))

$$\mathbf{e}(\xi): 0 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 0$$

of (\mathbb{R}, A) -Lie algebras.

Lemma 3.18. *The algebra of L -invariants $\text{Hom}_A(\Sigma'_A[s^2L'], A)^L$ is as a graded commutative algebra over the reals isomorphic to the G -invariants $(\text{Hom}_{\mathbb{R}}(\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}], \mathbb{R}))^G$, with G -action on $\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}]$ induced by the adjoint representation. Consequently, when G is connected, the algebra of L -invariants $\text{Hom}_A(\Sigma'_A[s^2L'], A)^L$ is as a graded commutative algebra over the reals isomorphic to the \mathfrak{g} -invariants $(\text{Hom}_{\mathbb{R}}(\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}], \mathbb{R}))^{\mathfrak{g}}$, with \mathfrak{g} -action on $\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}]$ induced by the adjoint representation of \mathfrak{g} on itself.*

In particular, the algebra of L -invariants $\text{Hom}_A(\Sigma'_A[s^2L'], A)^L$ depends only on G and the adjoint action (and not on the specific extension $\mathbf{e}(\xi)$ of Lie-Rinehart algebras, that is, not explicitly on L); further, the algebra $\text{Hom}_{\mathbb{R}}(\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}], \mathbb{R})$ is the algebra $\mathbb{R}[c_1, \dots, c_m]$ of polynomials on a basis $\{c_1, \dots, c_m\}$ of $\text{Hom}_{\mathbb{R}}([s^2\mathfrak{g}], \mathbb{R})$, and hence the algebra of L -invariants $\text{Hom}_A(\Sigma'_A[s^2L'], A)^L$ is as a graded commutative algebra over the reals isomorphic to the G -invariants $\mathbb{R}[c_1, \dots, c_m]^G$ and thence, when G is connected, to the \mathfrak{g} -invariants $\mathbb{R}[c_1, \dots, c_m]^{\mathfrak{g}}$. Thus we see that $\text{Hom}_A(\Sigma'_A[s^2L'], A)^L$ is isomorphic to the algebra of G -invariants of the algebra of polynomial functions on \mathfrak{g} with respect to an \mathbb{R} -basis, that is, under the present circumstances, the source of our Chern-Weil map (3.8.2) already looks like an appropriate classical object.

Proof of (3.18). Write $B = C^\infty(P)$, $L_N = \Gamma(\tau_N)$, and $L_B = \Gamma(\tau_P)$. By construction, in view of what was said about Atiyah sequences in (2.2), $L = E(\xi)$ coincides with the invariants L_B^G . Since ξ is a principal bundle for $\text{ad}(\xi)$, as B -modules, the induced module $B \otimes_A L'$ is that of sections of the induced bundle $\xi^*(\text{ad}(\xi))$ and hence isomorphic to $B \otimes_{\mathbb{R}} \mathfrak{g}$; under this isomorphism, the G -module structure on $B \otimes_A L'$ coming from the G -action on the first factor B corresponds to the diagonal action of G on $B \otimes_{\mathbb{R}} \mathfrak{g}$. Furthermore, $\text{Hom}_A(\Sigma'_A[s^2L'], B)$ being endowed with the obvious G -action induced by the G -action on B , the obvious map

$$\text{Hom}_A(\Sigma'_A[s^2L'], A) \longrightarrow \text{Hom}_A(\Sigma'_A[s^2L'], B)^G$$

into the G -invariants is an isomorphism. On the other hand, we have a chain

$$\begin{aligned} \text{Hom}_A(\Sigma'_A[s^2L'], B) &\cong \text{Hom}_B(B \otimes_A \Sigma'_A[s^2L'], B) \\ &\cong \text{Hom}_B(\Sigma'_B[B \otimes_A s^2L'], B) \\ &\cong \text{Hom}_B(\Sigma'_B[B \otimes s^2\mathfrak{g}], B) \\ &\cong \text{Hom}_{\mathbb{R}}(\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}], B) \end{aligned}$$

of obvious isomorphisms of graded \mathbb{R} -algebras and, in view of what has been said before, the resulting isomorphism

$$\text{Hom}_A(\Sigma'_A[s^2L'], B) \longrightarrow \text{Hom}_{\mathbb{R}}(\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}], B)$$

is one of G -modules, provided we take the G -structure on $\text{Hom}_{\mathbb{R}}(\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}], B)$ given by

$$(y, \phi) \longmapsto y(\phi) = y \circ \phi \circ \text{Ad}(y^{-1}), \quad y \in G, \quad \phi \in \text{Hom}_{\mathbb{R}}(\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}], B),$$

where, with an abuse of notation, the symbol Ad refers to the G -action on $\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}]$ induced by the adjoint representation of G on \mathfrak{g} . Furthermore, the isomorphic objects

$$\text{Hom}_B(B \otimes_A \Sigma'_A[s^2L'], B), \quad \text{Hom}_B(\Sigma'_B[B \otimes_A s^2L'], B), \quad \text{Hom}_B(\Sigma'_B[B \otimes s^2\mathfrak{g}], B)$$

admit structures of (B, L_B) -modules and of G -modules in such a way that, if we write M for any of these, we have

$$(3.19) \quad (M^G)^L = (M^{L_B})^G.$$

In view of (3.19), we conclude that

$$\begin{aligned} \mathrm{Hom}_A(\Sigma'_A[s^2L'], A)^L &\cong (\mathrm{Hom}_B(\Sigma'_B[B \otimes s^2\mathfrak{g}], B)^G)^L \\ &\cong (\mathrm{Hom}_B(\Sigma'_B[B \otimes s^2\mathfrak{g}], B)^{L_B})^G \\ &\cong (\mathrm{Hom}_{\mathbb{R}}(\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}], B^{L_B}))^G \\ &\cong (\mathrm{Hom}_{\mathbb{R}}(\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}], \mathbb{R}))^G. \quad \square \end{aligned}$$

To explain the formal nature of our argument relating our Chern-Weil map (3.8.1) with the classical one, we momentarily return to an arbitrary ground ring R , arbitrary commutative R -algebra A , and arbitrary extension (2.1) of Lie-Rinehart algebras, subject only to the condition that the extension split in the category of A -modules. We then consider the graded A -algebra $\Sigma_A^*[s^2L'] = \mathrm{Hom}_A(\Sigma_A[s^2L'], A)$ of A -multilinear symmetric functions on s^2L' with values in A , with the shuffle product as multiplication; cf. what is said in (1.6.4) above. For $\alpha \in L$ and $\phi \in \mathrm{Hom}_A(\Sigma_A[s^2L'], A)$, let

$$(\zeta(\alpha))(\phi) = \phi \circ (\mathrm{ad}(\alpha)) - \alpha \circ \phi: \Sigma_A[s^2L'] \longrightarrow A;$$

here $(\alpha \circ \phi)(x) = \alpha(\phi(x))$ where, with an abuse of notation, the operator on A induced by α is denoted by α as well. Inspection shows that this induces an action

$$L \rightarrow \mathrm{End}(\Sigma_A^*[s^2L'])$$

of L on $\Sigma_A^*[s^2L']$. The morphism (1.6.4.13) of graded A -algebras now looks like

$$(3.20) \quad \mathrm{Hom}_A(\Sigma'_A[s^2L'], A) \longrightarrow \Sigma_A^*[s^2L']$$

and is compatible with the L -structures and natural in the data in the appropriate sense; when L' is free, this map sends a polynomial function in a basis of L' to the corresponding symmetric function on L' , cf. what is said in (1.6.4) above.

We now suppose that the ground ring R contains the rationals so that (3.20) is an isomorphism. Then the composite of the inverse of (3.20), restricted to the L -invariants, with the Chern-Weil map (3.1.1), yields a morphism

$$(3.21) \quad (\Sigma_A^*[s^2L'])^L \rightarrow \mathrm{Alt}_A(L'', A)$$

of differential graded A -algebras which is defined on the invariant symmetric functions and coincides formally with the classical Chern-Weil map [7, 17]. Under the circumstances spelled out just before (3.18), in view of the statement of (3.18), we have

$$(\Sigma_A^*[s^2L'])^L \cong \mathrm{Inv}(\mathfrak{g}) = (\Sigma_{\mathbb{R}}^*[s^2\mathfrak{g}])^G$$

and, keeping in mind that

$$H^*(\text{Alt}_A(L'', A)) \cong H_{\text{deRham}}^*(N, \mathbb{R}),$$

our Chern-Weil map (3.8.1) boils indeed down to the classical one

$$\text{Inv}(\mathfrak{g}) \longrightarrow H_{\text{deRham}}^*(N, \mathbb{R}),$$

perhaps up to certain multiplicative factors (involving the factorials) which are due to the different possibilities of defining the de Rham algebra in characteristic zero and to the definition of the Chern-Weil map directly in terms of the algebra of symmetric functions. We note that the description of the Chern-Weil map in [25] involves actually the algebra of polynomial functions (on \mathfrak{g}) and not that of symmetric functions and is considerably closer to ours than that in the other sources [7, 17].

4. Examples

(1) We take the ground ring R to be that of the reals \mathbb{R} . Let N be a smooth finite dimensional manifold, write $A = C^\infty(N)$, let $L'' = \text{Vect}(N)$, the (\mathbb{R}, A) -Lie algebra of smooth vector fields on N , and let L' be the (\mathbb{R}, A) -Lie algebra which as an A -module is just A , with trivial (= abelian) Lie algebra structure. Then, on the one hand, $H^2(\text{Alt}_A(L'', A))$ is isomorphic to $H_{\text{deRham}}^2(N, \mathbb{R})$, see [33] for details while, on the other hand, in view of Theorem 2.6, the cohomology group $H^2(\text{Alt}_A(L'', A))$ classifies extensions of (\mathbb{R}, A) -Lie algebras of the kind

$$(4.1) \quad \mathbf{e}: 0 \rightarrow A \rightarrow L \rightarrow L'' \rightarrow 0$$

subject to the requirement that the adjoint representation of L on itself induces the L'' -module structure on A . Under these circumstances our Chern-Weil map (3.8.2) has a very simple form. Indeed, as an A -module, L' may be written as an induced module $L' = A \otimes_{\mathbb{R}} \mathbb{R}$, and the same is manifestly true of the graded symmetric coalgebra $\Sigma'_A[s^2 L']$ which is in fact isomorphic to $A \otimes_{\mathbb{R}} \Sigma'_{\mathbb{R}}[s^2 \mathbb{R}]$. Hence the graded commutative algebra $\text{Hom}_A(\Sigma'_A[s^2 L'], A)$ may be rewritten $\text{Hom}_{\mathbb{R}}(\Sigma'_{\mathbb{R}}[s^2 \mathbb{R}], A)$, and hence the subalgebra of invariants $\text{Hom}_A(\Sigma'_A[s^2 L'], A)^L$ looks like $\text{Hom}_{\mathbb{R}}(\Sigma'_{\mathbb{R}}[s^2 \mathbb{R}], \mathbb{R})$, which is just the polynomial algebra $\mathbb{R}[c]$ on a basis $\{c\}$ of $\text{Hom}_{\mathbb{R}}(s^2 \mathbb{R}, \mathbb{R})$. Consequently our Chern-Weil map (3.8.2) looks like

$$(4.2) \quad \mathbb{R}[c] \longrightarrow H_{\text{deRham}}^{2*}(N, \mathbb{R}).$$

When the extension \mathbf{e} does not come from a principal S^1 -bundle, that is to say, when the class $[\Omega] \in H_{\text{deRham}}^2(N, \mathbb{R})$ is not integral (i. e. does not have integral periods), this kind of example is not covered by the classical theory.

(2) More generally, let $\xi: P \rightarrow N$ be a principal bundle, with structure group G and Lie algebra \mathfrak{g} , let $L' = \mathfrak{g}(\xi) = \Gamma(\text{ad}(\xi))$ be the A -Lie algebra, viewed as an (\mathbb{R}, A) -Lie algebra with trivial action on A , which as an A -module is the space of sections of the adjoint bundle $\text{ad}(\xi)$, and consider an extension

$$(4.3) \quad \mathbf{e}: 0 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 0$$

of Lie-Rinehart algebras having the same outer action $L'' \rightarrow \text{ODer}(L')$ as that coming from the extension

$$\mathbf{e}(\xi): 0 \rightarrow L' \rightarrow E(\xi) \rightarrow L'' \rightarrow 0$$

given in (2.2.2); we remind the reader that the notion of outer action has been reproduced in Section 2. Under these circumstances, we can still conclude that the subalgebra of invariants $\text{Hom}_A(\Sigma'_A[s^2L'], A)^L$ looks like $\text{Hom}_{\mathbb{R}}(\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}], \mathbb{R})^G$, that is to say, it is the algebra of G -invariants of the algebra $\mathbb{R}[c_1, c_2, \dots, c_m]$ of polynomial functions on \mathfrak{g} with respect to an \mathbb{R} -basis $\{c_1, c_2, \dots, c_m\}$ of $\text{Hom}_{\mathbb{R}}(s^2\mathfrak{g}, \mathbb{R})$. In fact, the L -action on $\text{Hom}_A(\Sigma'_A[s^2L'], A)$ passes to an L'' -action on the algebra of invariants $\text{Hom}_A(\Sigma'_A[s^2L'], A)^{L'}$, determined entirely by the corresponding outer action $L'' \rightarrow \text{ODer}(L')$, and the algebra of invariants $\text{Hom}_A(\Sigma'_A[s^2L'], A)^L$ may be rewritten

$$\text{Hom}_A(\Sigma'_A[s^2L'], A)^L = \left(\text{Hom}_A(\Sigma'_A[s^2L'], A)^{L'} \right)^{L''}.$$

However, the algebra of invariants $\left(\text{Hom}_A(\Sigma'_A[s^2L'], A)^{L'} \right)^{L''}$, in turn, may be rewritten

$$\left(\text{Hom}_A(\Sigma'_A[s^2L'], A)^{L'} \right)^{L''} = \text{Hom}_A(\Sigma'_A[s^2L'], A)^{E(\xi)},$$

and, by virtue of (3.18), we know that, as a graded commutative algebra over the reals, the algebra $\text{Hom}_A(\Sigma'_A[s^2L'], A)^{E(\xi)}$ is isomorphic to the algebra of invariants $\text{Hom}_{\mathbb{R}}(\Sigma'_{\mathbb{R}}[s^2\mathfrak{g}], \mathbb{R})^G$. Consequently our Chern-Weil map (3.8.2) looks like

$$(4.4) \quad \mathbb{R}[c_1, c_2, \dots, c_m]^G \longrightarrow H_{\text{deRham}}^{2*}(N, \mathbb{R}).$$

When the extension \mathbf{e} does not come from a principal G -bundle, again this kind of example is not covered by the classical theory. To obtain explicit examples, suppose that the Lie algebra \mathfrak{g} has a non-trivial centre \mathfrak{z} , and let $\zeta(\xi): P \times_G \mathfrak{z} \rightarrow N$ be the corresponding associated bundle with fibre \mathfrak{z} —this bundle is trivial when G is connected. When the cohomology group $H_{\text{deRham}}^2(N, \zeta(\xi))$ is non-zero, Theorem 2.7 provides a wealth of examples of extensions of L'' by L' which do not come from a principal bundle but have the same outer action of L'' on L' as that coming from the principal bundle ξ we started with. An approach to the corresponding global theory, phrased in terms of Lie groupoids, has been given in [23].

(3) Let N be a smooth finite dimensional manifold, let $C \subseteq N$ be a compact subset, not necessarily a smooth manifold, and let A_C be the algebra of smooth functions on C in the sense of WHITNEY [38], [39]. It will here be convenient to take for a *smooth function* f on C in this sense a class of smooth functions h defined on N , two functions being identified whenever they coincide on C . (It is also customary to take classes of smooth functions h defined only on a neighborhood of C in N .) Let I_C be the ideal of smooth functions on N that vanish on C , so that $A_C = C^\infty(N)/I_C$. Furthermore, let $\text{Vect}(N, C) \subseteq \text{Vect}(N)$ be the set of smooth vector fields X on N that preserve I_C in the sense that

$$Xh = 0 \quad \text{on } C \quad \text{whenever} \quad h = 0 \quad \text{on } C.$$

It is readily seen that $\text{Vect}(N, C)$ inherits a structure of an $(\mathbb{R}, C^\infty(N))$ -Lie algebra from $\text{Vect}(N)$. Let $L_C = A_C \otimes_{C^\infty(N)} \text{Vect}(N, C)$; inspection shows that the

$(\mathbb{R}, C^\infty(N))$ -Lie algebra structure on $\text{Vect}(N, C)$ passes to that of an (\mathbb{R}, A_C) -Lie algebra on L_C . In the special case where C is a smooth submanifold of N , the obvious map

$$\text{Vect}(N, C) \longrightarrow \text{Vect}(C)$$

induces an isomorphism

$$L_C \longrightarrow \text{Vect}(C).$$

Hence we refer to L_C as the (\mathbb{R}, A_C) -Lie algebra of *smooth vector fields on C* .

As in (2), write $A = C^\infty(N)$, let $\xi: P \rightarrow N$ be a principal bundle, with structure group G and Lie algebra \mathfrak{g} , and consider the A -Lie algebra $\mathfrak{g}(\xi)$, viewed as an (\mathbb{R}, A) -Lie algebra with trivial action on A . Let $\mathfrak{g}_C(\xi) = A_C \otimes_{C^\infty(N)} \mathfrak{g}(\xi)$; it is manifestly a projective A_C -module and inherits an obvious structure of an A_C -Lie algebra; it will henceforth be viewed as an (\mathbb{R}, A_C) -Lie algebra with trivial action on A_C . Under these circumstances, our Chern-Weil map (3.8.2) arising from an arbitrary extension

$$(4.5) \quad \mathbf{e}: 0 \rightarrow \mathfrak{g}_C(\xi) \rightarrow L \rightarrow L_C \rightarrow 0$$

of Lie-Rinehart algebras looks like

$$(4.6) \quad \text{Hom}_{A_C}(\Sigma'_{A_C}[s^2 \mathfrak{g}_C(\xi)], A_C)^L \longrightarrow H_{A_C}^{2*}(\text{Alt}_{A_C}(L_C, A_C)).$$

Again this kind of examples is not covered by the classical approach. In particular, we may restrict the corresponding sequence (2.2.2) to the $(\mathbb{R}, C^\infty(N))$ -Lie algebra $\text{Vect}(N, C)$, as indicated in the commutative diagram

$$(4.7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{g}(\xi) & \longrightarrow & E(\xi, C) & \longrightarrow & \text{Vect}(N, C) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{g}(\xi) & \longrightarrow & E(\xi) & \longrightarrow & \text{Vect}(N) \longrightarrow 0 \end{array}$$

where the $(\mathbb{R}, C^\infty(N))$ -Lie algebra $E(\xi, C)$ is defined by the requirement that $E(\xi, C)$, $E(\xi)$, $\text{Vect}(N, C)$, and $\text{Vect}(N)$ constitute a pull back diagram. Since, as $(C^\infty(N))$ -modules, the bottom row of (4.8) splits, so does the top row; consequently, with the notation $E_C(\xi) = A_C \otimes_{C^\infty(N)} E(\xi, C)$, the corresponding sequence

$$(4.8) \quad 0 \longrightarrow \mathfrak{g}_C(\xi) \longrightarrow E_C(\xi) \longrightarrow L_C \longrightarrow 0$$

is still exact and in particular an extension of (\mathbb{R}, A_C) -Lie algebras. Our Chern-Weil map (4.6) furnishes characteristic classes for it.

More generally, we now consider a general extension of Lie-Rinehart algebras of the kind (4.5), subject only to the condition that its corresponding outer action

$$L_C \longrightarrow \text{ODer}(\mathfrak{g}_C(\xi)),$$

cf. Section 2, coincides with that for the extension (4.8). The argument already used before shows that the algebra of invariants $\text{Hom}_{A_C}(\Sigma'_{A_C}[s^2 \mathfrak{g}_C(\xi)], A_C)^L$ may be rewritten

$$\begin{aligned} \text{Hom}_{A_C}(\Sigma'_{A_C}[s^2 \mathfrak{g}_C(\xi)], A_C)^L &= \left(\text{Hom}_{A_C}(\Sigma'_{A_C}[s^2 \mathfrak{g}_C(\xi)], A_C)^{L'} \right)^{L_C}, \\ &= \text{Hom}_{A_C}(\Sigma'_{A_C}[s^2 \mathfrak{g}_C(\xi)], A_C)^{E_C(\xi)}, \end{aligned}$$

where $L' = \mathfrak{g}_C(\xi)$, and hence does not depend on the extension (4.5) but only on the corresponding outer action. Furthermore, restriction induces a commutative diagram

$$(4.9) \quad \begin{array}{ccc} \mathbb{R}[c_1, c_2, \dots, c_m]^G & \longrightarrow & H_{\text{deRham}}^{2*}(N, \mathbb{R}) \\ \downarrow & & \downarrow \\ \text{Hom}_{A_C}(\Sigma'_{A_C}[s^2 \mathfrak{g}_C(\xi)], A_C)^{E_C(\xi)} & \longrightarrow & H^{2*}(\text{Alt}_{A_C}(L_C, A_C)), \end{array}$$

the horizontal arrows being the corresponding Chern-Weil maps.

This discussion raises, among others, the following two questions:

(i) Is the vertical morphism

$$\mathbb{R}[c_1, c_2, \dots, c_m]^G \longrightarrow \text{Hom}_{A_C}(\Sigma'_{A_C}[s^2 \mathfrak{g}_C(\xi)], A_C)^{E_C(\xi)}$$

in (4.9) an isomorphism?

(ii) Can we intrinsically define the notion of a smooth principal bundle for A_C merely over C which is *not* necessarily induced from a principal bundle over N ?

(iii) If the answer to (2) is yes, can we then determine the corresponding algebra of invariants directly in terms of the new structure over C ?

(4) Let \mathcal{F} be a *transversally complete* foliation of a smooth manifold V [1], write $\tau_{\mathcal{F}}: T\mathcal{F} \rightarrow V$ for the tangent bundle of \mathcal{F} , and let $\nu_{\mathcal{F}}: Q \rightarrow V$ be its normal bundle, so that $Q = TV/T\mathcal{F}$. Let $E(\mathcal{F})$ be the Lie algebra of vector fields on V preserving the foliation, and let A be the algebra of smooth functions on the leaf space, i. e. smooth functions on V which are constant on the leaves. Then, with the obvious structure, the pair $(A, E(\mathcal{F}))$ constitutes a Lie-Rinehart algebra. Since \mathcal{F} is transversally complete, the closures of the leaves constitute a smooth fibre bundle $F \rightarrow V \rightarrow W$, and the algebra A may be identified with the algebra of smooth functions on W ; moreover, the obvious map from $E(\mathcal{F})$ to $\text{Vect}(W)$ which is part of the Lie-Rinehart structure of $(A, E(\mathcal{F}))$ is surjective, and there results an extension

$$(4.10) \quad \mathbf{e}_{\mathcal{F}}: 0 \rightarrow L' \rightarrow E(\mathcal{F}) \rightarrow \text{Vect}(W) \rightarrow 0$$

of Lie-Rinehart algebras. The kernel L' is in fact the space of sections of a Lie algebra bundle on W . Our Chern-Weil construction yields characteristic classes in $H_{\text{deRham}}^{2*}(W, \mathbb{R})$ for this extension.

We conclude with an illustration which I learnt from A. Weinstein: Let $V = \text{SU}(2) \times \text{SU}(2)$, and let \mathcal{F} be the foliation defined by a dense one-parameter subgroup in the maximal torus $S^1 \times S^1$ in $\text{SU}(2) \times \text{SU}(2)$. Then the space W is $S^2 \times S^2$, and the Chern-Weil construction yields a characteristic class in $H_{\text{deRham}}^2(S^2 \times S^2, \mathbb{R})$ which may be viewed as an *irrational Chern class*. In view of a result of ALMEIDA AND MOLINO [1], the transitive Lie algebroid corresponding to (4.10) does not integrate to a principal bundle; in fact, MACKENZIE's integrability obstruction [21] is non-zero. It is clear that there are many other examples of this kind.

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